Brownian motion with the exposure time control

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We solve a classical analog of the quantum traversal time problem. The classical diffusion equation is modified to control the amount of time spent by a particle in a specified region of space. The ''clocked'' diffusion equation is solved for an ensemble of Brownian particles confined in a closed volume. Long term behavior of the exposure time distribution is analyzed. A general recipe for including time control into fieldlike equations of motion describing evolution of a physical system is proposed. [S1063-651X(96)11206-X]

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I. INTRODUCTION

Over the years, there has been considerable interest in the general problem of determining the amount of time, τ, a physical system, whose motion is described by partial differential equations rather than in terms of classical trajectories, spends in a specified region of space Ω. Until recently, the attention was focused on quantum mechanics, in particular, on the tunneling time problem (for reviews, see [1]). In Refs. [2,3] similar techniques were applied in order to understand the occurrence of superluminal velocities in the propagation of classical electromagnetic waves [4,5]. In this paper we extend the approach to classical stochastic systems and, in particular, to the case of Brownian motion. Common to all mentioned problems is the apparent absence of any information about τ in the initial (Schrödinger, Maxwell, diffusion, etc.) equation of motion. We will provide a simple general recipe for building in the exposure (traversal) time control and solve the ''clocked'' diffusion equation for Brownian motion in a closed volume. The paper is organized as follows. In Sec. II we introduce a clocked diffusion equation describing the distribution of exposure times, $Q(x,t|\tau)$, for an ensemble of Brownian particles. In Sec. III we formally solve the equation by expanding $Q(x,t|\tau)$ in the eigenfunctions of the corresponding eigenvalue problem. In Sec. IV we investigate analytical behavior of the eigenvalues E_n and eigenfunctions ϕ_n. In Sec. V we demonstrate that the analytical properties of E_n and ϕ_n impose correct behavior on $Q(x,t|\tau)$ and obtain for the latter a simple series representation. In Sec. VI we use the steepest descent method to study the long time behavior of $Q(x,t|\tau)$. We also provide a numerical test of the theory. Section VII contains our conclusions.

II. EXPOSURE TIME CONTROL AND THE CLOCKED DIFFUSION EQUATION

Consider an ensemble of Brownian particles confined in a (one dimensional) volume $[-L,L]$, e.g., put in a plugged test tube of a length $2L$. The concentration of particles, $Q(x,t)$, satisfies the diffusion equation [6]

$$\frac{\partial Q(x,t)}{\partial t}=D\frac{\partial^2 Q(x,t)}{\partial x^2}, \tag{2.1}$$

where D is the diffusion coefficient. For a given initial distribution $Q(x,0)$, Eq. (2.1) allows one to determine the concentration $Q(x,t)$ at any $t>0$. However, a more detailed analysis may be required. Let a region $[-a,a]$ inside the tube be exposed to the light or radiation. One can think of several ways in which physical properties of the particles may be affected by exposure to radiation or light. If, for example, a particle is coated in photoemulsion, the light will darken its color in proportion to the time it spends in the illuminated region. For live bacteria moving chaotically in a liquid medium the length of exposure to harmful radiation will determine whether a bacterium is healthy, sick, or dead. In both bases, full statistical information about the state of the particle is contained in the concentration $Q(x,t|\tau)$ yielding the density of the particles which have in the past, i.e., prior to t, spent in $[-a,a]$ a net duration τ.

Note first that the diffusion equation (2.1) gives no clue as to the length of time a particle spends inside $[-a,a]$. Indeed, Brownian particles can travel along various paths, each having a different value of τ. In order to construct $Q(x,t)$ one adds up probabilities for all possible paths ending in x at time t, [6] thus obliterating all information about τ. It would appear then that in order to build the exposure time control into the classical Brownian motion one has to monitor the behavior of the Brownian paths for all $t'<t$ which will preclude the description of $Q(x,t|\tau)$ in terms of a simple partial differential equation similar to Eq. (2.1). Fortunately, this is not the case. In fact, $Q(x,t|\tau)$ satisfies the clocked diffusion equation [7] ($\theta_{xy}(z)=1$, $x\leqslant z\leqslant y$, and 0 otherwise)

$$\frac{\partial Q(x,t|\tau)}{\partial t}=D\frac{\partial^2 Q(x,t|\tau)}{\partial x^2}-\theta_{-aa}(x)\frac{\partial Q(x,t|\tau)}{\partial \tau}. \tag{2.2}$$

A rigorous derivation of (2.2) based on the Wiener integral has been given in Ref. [7]. However, a simpler recipe for constructing Eq. (2.2) is available. Indeed, a particle can leave (enter) a phase volume $dxd\tau$ either by diffusing to a different location, as described by the first term on the right-hand side (rhs) of Eq. (2.2), or, provided it is inside $[-a,a]$, by increasing its exposure time τ. The second possibility accounts for the second term on the rhs of Eq. (2.2). We can put $D=1$ in Eq. (2.2) by introducing dimensionless variables $x\rightarrow x/L$, $t\rightarrow t/t_0$, and $\tau\rightarrow\tau/t_0$, where $t_0=L^2/D$ is the mean time it takes a Brownian particle to travel the distance L. The scaled variables will be used unless stated otherwise.

Next we will solve Eq. (2.2), assuming that the light is switched on at $t=0$ when the concentration $Q(x,0)$ is in equilibrium, i.e., the particles are uniformly distributed throughout the volume. Thus we have the initial condition.

$$Q(x,0|\tau)=\delta(\tau) \tag{2.3}$$

together with the boundary conditions

$$Q'(1,t|\tau)=Q'(-1,t|\tau)=0, \tag{2.4}$$

imposing zero flux at the ends of the test tube. In Eq. (2.3) $\delta(x)$ is the Dirac δ function and the prime denotes differentiation with respect to x.

Note finally that the system we have chosen is in equilibrium at the lowest macroscopical level of description: the constant concentration $Q(x,t)$ will remain unchanged at all times. However, underlying this evolution (or rather the nonevolution) of the total concentration is the microscopic Brownian motion which carriers the particles across the test tube. This is revealed if a more detailed description of the system is provided. Indeed, let us return to the model in which the particle's color gets darker in proportion to the exposure time. Assuming the particles are all white at $t=0$, for $t>0$ we should see the dark color spread from the illuminated region until after a sufficiently long time the whole test tube is colored black. This is the nonequilibrium evolution we seek to describe with the help of Eq. (2.2).

III. EIGENFUNCTION EXPANSION FOR $Q(x,t|\tau)$

The Fourier transform of $Q(x,t|\tau)$,

$$Q(x,t|V)=\int_{-\infty}^{\infty}Q(x,t|\tau)\exp(iV\tau)d\tau \tag{3.1}$$

satisfies a diffusion equation (2.1) with an additional potential term $iV\ \theta_{-\beta\beta}(x)\ Q(x,t|V)$,

$$\frac{\partial Q(x,t|V)}{\partial t}=\frac{\partial^2 Q(x,t|V)}{\partial x^2}+iV\theta_{-\beta\beta}(x)Q(x,t|V), \tag{3.2}$$

where

$$\beta\equiv a/L. \tag{3.3}$$

Although non-Hermitian, the operator on the rhs of Eq. (3.2) has a complete set of eigenfunctions for each value of V. Using these sets of expand $Q(x,t|V)$, inverting the Fourier transform (3.1), and replacing V by iW we obtain the solution of Eq. (2.2) satisfying conditions (2.3) and (2.4)

$$Q(x,t|\tau)=\frac{1}{2\pi}\sum_{n=0}^{\infty}\int_{-i\infty}^{i\infty}\exp[W\tau-E_n(W)t]\times(1|\phi_n(W))\phi_n(W,x)dW. \tag{3.4}$$

In Eq. (3.4) $(f|g)$ denotes a scalar product without conjugation, $(g|f)\equiv\int_{-1}^{1}f(x)g(x)dx$, and $\phi_n(W,x)$ are the normalized solutions of the Sturm-Liouville problem

$$-\phi_n''(W,x)+W\theta_{-\beta\beta}(x)\phi_n(W,x)=E_n(W)\phi_n(W,x),\quad n=0,1,2\ldots, \tag{3.5}$$

$$\phi_n'(W,1)=\phi_n'(W,-1)=0,$$

$$(\phi_m(W)|\phi_n(W))=\delta_{mn}.$$

Symmetric eigenstates in Eq. (3.4), $\phi_n(W,x)=\phi_n(W,-x)$, are explicitly given by

$$\phi_n(W,x)=\Phi[E_n(W),W,x]/N[E_n(W),W]^{1/2}, \tag{3.6}$$

where

$$\begin{aligned}\Phi(E,W,x)&=\cos(k_1\beta)\cos[k(1-x)],\quad 0<x<\beta\\&=\cos[k(1-\beta)]\cos(k_1x),\quad \beta<x<1,\end{aligned} \tag{3.7}$$

$$k(E)\equiv E^{1/2},$$

$$k_1(E,W)\equiv(E-W)^{1/2},$$

and

$$N(E,W)=(\Phi|\Phi). \tag{3.8}$$

The corresponding eigenvalues $E_n(W)$, $n=0,1,2\ldots$, are the roots of the transcendental equation

$$F(E_n,W)=0,$$

$$F(E,W)\equiv k\,tg[k(1-\beta)]+k_1 tg(k_1\beta). \tag{3.9}$$

Next we evaluate the distribution in Eq. (3.4).

IV. COMPLEX DEGENERACIES AND AVOIDED CROSSINGS

To proceed, we require the properties of $E_n(W)$ and $\phi_n(W,x)$ in the entire complex plane of parameter W. It is convenient to introduce a single valued function $\mathcal{E}(W)$ defined on a Riemann surface $\mathcal{R}$ such that, for a specified W, $E_n(W)$ is given by the value of $\mathcal{E}(W)$ on the nth ($n=0,1,2\ldots,$) sheet $\mathcal{R}$. Apart from the boundary condition, Eq. (3.5) looks remarkably like the stationary Schrödinger equation for a particle in a box with additional rectangular potential of magnitude W. A quantum-mechanical analogy is indeed helpful. As in quantum mechanics, the ''levels'' $E_n(W)$ in Eq. (3.5) cannot cross for real values of W, but complex degeneracies are allowed. The nature of these degeneracies can be understood as follows: on the edges of the complex W plane, excluding the real axis, ($W=|W|\exp(i\Phi)$, $|W|\to\infty$, $\Phi\neq 0,\pi$), the eigenvalues

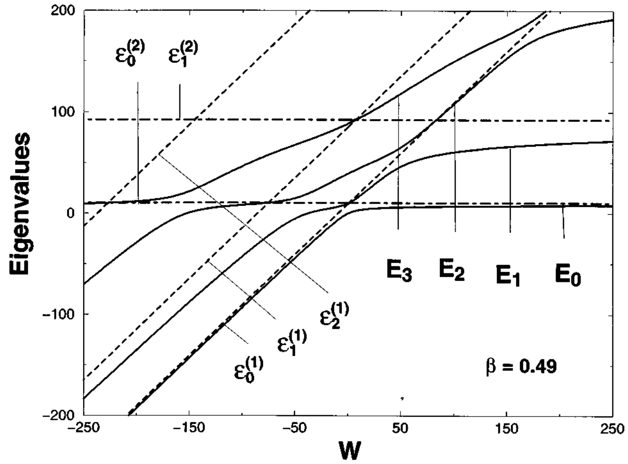

FIG. 1. Eigenvalues of the Sturm-Liouville problem (3.5) E_n, $n=0,1,2,3$ vs (real) W for $a/L=0.49$ (solid). Also shown are (dashed) $\varepsilon_1^{(1)}$, $i=0,1,2$ and (dot-dashed) $\varepsilon_j^{(2)}$, $j=0,2$, defined in Eqs. (4.1) and (4.2).

$E_n(W)$ and eigenstates $\Phi_n(W,x)$ fall into two separate families:

$$\varepsilon_j^{(1)}(W)=\pi^2(2j+1)^2/4\beta^2+W, \tag{4.1}$$

$$\chi_j^{(1)}(x)=\theta_{-\beta\beta}(x)\beta^{-1/2}\cos[(2j+1)\pi x/2\beta],$$

$$j=0,1,2\ldots,$$

and

$$\varepsilon_j^{(2)}(W)=\pi^2(2j+1)^2/4(1-\beta)^2, j=0,1,2\ldots, \tag{4.2}$$

$$\chi_j^{(2)}(x) =\{\theta_{-1-\beta}(x)\cos[(2j+1)\pi(1+x)/2(1-\beta)] +\theta_{\beta 1}(x)\cos[(2j+1)\pi(1-x)/2(1-\beta)]\}/(1-\beta)^{1/2}.$$

The states $\chi_j^{(1)}$ corresponding to levels $\varepsilon_j^{(1)}$ are ''quantized'' between the edges of the complex ''potential'' $W\theta_{-\beta\beta}(x)$, $-a/L<x<a/L$, their energies increasing with W. The states $\chi_j^{(2)}$ corresponding to $\varepsilon_j^{(2)}$ span the regions $[-1,-a/L]$ and $[a/L,1]$, their energies independent of W as $|W|\to\infty$. When applied for *all* values of W, Eqs. (4.1) and (4.2) predict that levels $\varepsilon_m^{(1)}$ and $\varepsilon_n^{(2)}$ would cross for real $W=W_{mn}$,

$$W_{mn}=\pi^2(2n+1)^2L^2/4a^2-\pi^2(2m+1)^2L^2/4(L-a)^2,$$
$$m,n=0,1,2\ldots. \tag{4.3}$$

Of course, $\varepsilon_m^{(1)}$ and $\varepsilon_n^{(2)}$ are not valid eigenvalues for finite values of W. Rather, the crossings (4.3) are turned into avoided crossings [8,9]. Associated with an avoided crossing labeled (m,n) is a pair of complex conjugate points, $\mathcal{W}_{mn}$ and $\mathcal{W}_{mn}^*$ where the eigenvalues $E_m(W)$ and $E_n(W)$ become degenerate, e.g. $E_m(\mathcal{W}_{mn})=E_n(\mathcal{W}_{mn})\equiv\mathcal{E}_{mn}$ The eigenvalues $E_n(W)$, $n=0.1,2,3$ are shown in Fig. 1 for real W and

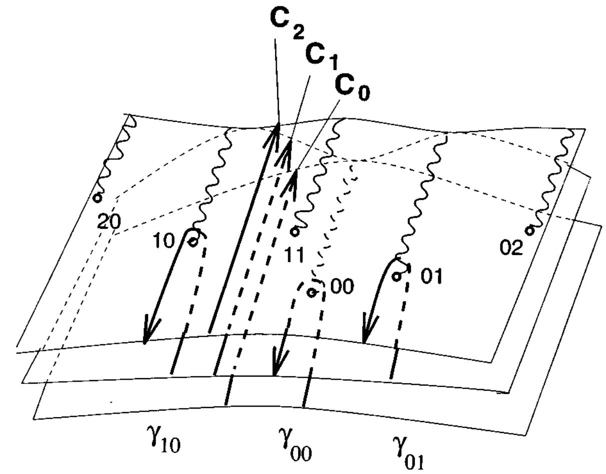

FIG. 2. The three sheets of the Riemann surface $\mathcal{R}$ shown for Im$W>0$. The sheets are joined at the branching points $\mathcal{W}_{mn}$ (small circles) with cuts indicated by wavy lines. Also shown are the integration contours in Eq. (3.4) for $n=0,1,2$ (C_0,C_1,C_2) and the loop contours γ_{00}, γ_{01}, and γ_{10} used in Eqs. (5.2).

$\beta=0.49$ together with $\varepsilon_i^{(1)}$, $i=0,1,2$ and $\varepsilon_j^{(2)}$, $j=0,1$. Note that the avoided crossings are best pronounced for $n=0$ and $n=1$.

The values $\mathcal{W}_{mn}$ and $\mathcal{E}_{mn}$ must be found numerically as the roots of two simultaneous transcendental equations,

$$F(\mathcal{E}_{mn},\mathcal{W}_{mn})=0, \tag{4.4}$$
$$\frac{\partial F(\mathcal{E}_{mn},\mathcal{W}_{mn})}{\partial E}=0, \quad m,n=0,1,2,\ldots,$$

where $F(E,W)$ is given by Eq. (3.9). Behavior of the $E_m(W)$ is known from quantum mechanics [8,9]. Namely, in the vicinity of W_{mn} we have

$$E_m(W)-E_n(W)\approx(W-\mathcal{W}_{mn})^{1/2}. \tag{4.5}$$

Thus, the Riemann surface $\mathcal{R}$ of $\mathcal{E}(W)$ consists of an infinite number of sheets joined pairwise at the branching points. The first three sheets with branch cuts chosen to run to infinity parallel to the imaginary axis are shown in Fig. 2 for Im$W>0$. Finally, using Eqs. (3.6) and (3.7) we define the eigenstates globally on $\mathcal{R}$,

$$\phi(W,x)\equiv\Phi[\mathcal{E}(W),W,x]/N[\mathcal{E}(W),W]. \tag{4.6}$$

As in quantum mechanics [8,9], eigenstates have quartic root singularities near complex branching points, so that for $W\to W_{mn}$ we have

$$\phi(W,x)\approx(W-\mathcal{W}_{mn})^{-1/4}. \tag{4.7}$$

V. SERIES REPRESENTATION FOR $Q(x,t|\tau)$

Now we can proceed with the evaluation of the exposure time distribution (3.4). Note first that since the exposure time τ is a non-negative quantity and cannot exceed the elapsed time t, distribution $Q(x,t|\tau)$ must vanish identically for $\tau<0$ and $\tau>t$. To show that this is indeed the case we start with the $n=0$ term in the sum (3.4). The integration contour

C_0 runs up the imaginary W axis on the first sheet of the Riemann surface (Fig. 2). Let the branch cut lie to the right of C_0 as shown in Fig. 1 and consider first $\tau<0$. As $|W|\to\infty$, on C_0 we have $E_0(W)\to\pi^2/4\beta^2+W$. Inspection of the exponent in Eq. (3.4) shows that the contour can be closed in the right half-plane and then transformed into a set of loop contours encircling the branching points $\mathcal{W}_{0n}$ and $\mathcal{W}^*_{0n}$, $n=0,1,2\ldots$, as shown in Fig. 2 for $n=0$. Consider next the second term in Eq. (3.5) and the contour C_1 shown in Fig. 2. On C_1, for $|W|\to\infty$, we have $E_1(W)=\pi^2/4(1-\beta)^2$ and again the contour can be transformed into a single loop encircling $\mathcal{W}_{00}$ and $\mathcal{W}^*_{00}$ but in the direction opposite to that shown in Fig. 2. Repeating this analysis for all terms in Eq. (3.4) we find two contours encircling each pair of branching points in opposite directions so that the sum is identically zero. Similarly, for $\tau>t$, all integration contours in Eq. (3.4) can be closed in the left half-plane and $Q(x,t|\tau)$ vanishes. For $0<\tau<t$, however, those contours on which $E_n(W)\to\text{const}$ for $|W|\to\infty$ can be closed in the left half-plane, whereas those on which $E_n(W)\to\text{const}+W$ must be closed in the right half of the W plane. As a result, for $0<\tau<t$, $Q(x,t|\tau)$ is given by the sum of all integrals along the loop contours γ_{mn}, $m,n=0.1,2\ldots.$ Finally, evaluating the δ-function contributions arising because the integrands in Eq. (3.4) do not vanish for $|W|\to\infty$, we arrive at the following expression for $Q(x,t|\tau)$:

$$Q(x,t|\tau)=\begin{cases}\delta(\tau)\sum_{n=0}^{\infty}\exp[-\pi^2(2n+1)^2t/4(1-\beta)^2] \\ \times(1|\chi_n^{(2)})\chi_n^{(2)}(x)+\delta(\tau-t) \\ \times\sum_{n=0}^{\infty}\exp[-\pi^2(2N+1)^2t/4\beta^2] \\ \times(1|\chi_n^{(1)})\chi_n^{(1}(x)+\sum_{m;n=0}^{\infty}I_{mn}(x,t|\tau), \\ 0\leqslant\tau\leqslant t \\ 0,\quad\text{otherwise,}\end{cases} \tag{5.1}$$

where

$$I_{mn}(x,t|\tau)\equiv\frac{i}{2\pi}\int_{\gamma_{mn}}\exp[W\tau-\mathcal{E}(W)t] \times(1|\phi(W))\phi(W,x)dW. \tag{5.2}$$

Note that according to Eq. (4.7), the integrand of Eq. (5.2) has integrable root singularities at the branching points $\mathcal{W}_{00}$ and $\mathscr{W}^*_{00}$. Note also that the first and the second terms in Eq. (5.1) describe the concentrations of those particles which have not yet entered and left the region $[-a/L,a/L]$, and had therefore spent either no time at all there or exactly t, respectively. For large t these concentrations are seen to decay exponentially with their coordinate profiles given by the lowest eigenfunctions $\chi_0^{(1)}(x)$ and $\chi_0^{(2)}(x)$ in Eqs. (4.1) and (4.2).

VI. LONG TIME BEHAVIOR OF $Q(x,t|\tau)$

Finally we investigate the asymptotic behavior of $Q(x,t|\tau)$. The first term on the rhs of Eq. (2.2) leads to diffusive spreading of $Q(x,t|\tau)$ while the second term attempts to propagate the part of the distribution contained in the exposed region forward in the τ coordinate. For very short times, the diffusive term can be neglected. In the long term limit, $t\gg1$, as Brownian particles forget their initial positions the diffusive term is important. Next we will demonstrate that in this limit $Q(x,t|\tau)$ looses its dependence on x but retains a peak in the τ coordinate, which moves toward larger τ's. For $t\gg1$, integrals (5.2) can be evaluated by the steepest descent method [10] and a simple analysis shows that the main contribution comes from $I_{00}(x,t|\tau)$ with other terms giving only exponentially small corrections. Transforming γ_{00} into a steepest descent contour we have

$$Q(x,t|\tau)\approx[2\pi t|\partial^2E_0(W)/\partial W^2|]^{-1/2}\exp[W\tau-E_0(W)t] \times(1|\phi_0(W))\phi_0(W,x)|_{W=W(\tau,t)}, \tag{6.1}$$

where the (real) saddle point $W(\tau,t)$ is defined by

$$\frac{\partial E_0[W(\tau,t)]}{\partial W}=\tau/t. \tag{6.2}$$

The exponent in Eq. (6.1) reaches its maximum value at $W(t,\tau)=0$ and with the help of the relation

$$\partial E_0(W)/\partial W|_{W=0}=\int_{-\beta}^{\beta}\phi_0^2(W)dx|_{W=0}=a/L \tag{6.3}$$

we find $Q(x,t|\tau)$ peaked around $\tau=(a/L)t$. More precisely, with the help of Eq. (6.1) we find both the mean exposure time, $\overline{\tau}$, and its variance, σ, independent of x.

$$\overline{\tau(t)}=[\partial E_0(0)/\partial W]t=(a/L)t, \tag{6.4}$$

$$\sigma(t)=[|\partial^2E_0(0)/\partial W^2|t]^{1/2}. \tag{6.5}$$

Thus in the long time limit $Q(x,t|\tau)$ tends to a homogeneous distribution with a peak in the τ coordinate propagating forward in τ with velocity equal to the ratio of the volume of exposed region to the total volume of the system, β, while the width of the peak increases as $t^{1/2}$. Whereas the first result is expected for a random walk in a finite volume, the dependence of σ on the size on β is worth a further discussion. Differentiating Eq. (6.3) with respect to W and calculating $\partial\phi_0^2(W)/\partial W|_{W=0}$ using the perturbation theory we have

$$\sigma(t,\beta)=2\left(\sum_{m=1}^{\infty}\frac{\sin^2(\pi m\beta)}{(\pi m)^4}t\right)^{1/2}. \tag{6.6}$$

The variance $\sigma(t,\beta)$, shown in Fig. 3, vanishes for $\beta=0$ and $\beta=1$, when the exposure times for all particles equals exactly 0 and t, respectively, while the maximum spread in τ, $\sigma\approx0.21t^{1/2}$ is achieved for $\beta=\frac{1}{2}$ when the volume of the exposed region is half the total volume of the system.

To provide a numerical test, we return to the original (unscaled) x,t and τ in Eq. (2.2) and propagate an initial Gaussian distribution $Q(x,0|t)=\exp(-\alpha\tau^2)$, $a=200$, for

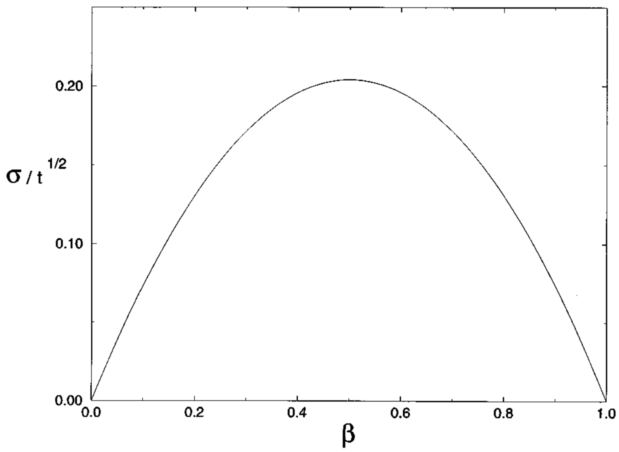

FIG. 3. Dependence of the variance σ on $\beta \equiv a/L$.

$\beta = 0.5$ and different values of the diffusion coefficient. The results ($t/t_0 = 0.01, 0.1, 1.0$) are shown in Figs. 4(a), and 4(b), and 4(c), respectively, with the asymptotic form (6.1) seen to be sufficiently accurate already for $t/t_0 \approx 1$. To summarize, at short times diffusion can be neglected and we see the particles inside $[-a,a]$ increase their exposure times whereas for the particles outside the region the τ's remain unchanged [Fig. 4(a)]. At intermediate times [Fig. 4(b)], diffusive mixing between the two regions smoothens the distribution which still shows a tendency for larger τ's inside and close to the exposed region. Finally, at large times the distribution looses its coordinate dependence and becomes homogeneous in x [Fig. 4(c)] . Note that the last result is not immediately obvious, as at any t one expects the particles currently inside the exposed region to have somewhat larger τ's than those outside it. However, in the large time limit, $t/t_0 \gg 1$, any such difference is lost against the rapidly growing width of the distribution σ.

VII. CONCLUSIONS

To conclude, we have solved the problem of determining the amount of time a Brownian particle spends in a specified region of space. Somewhat surprisingly, having started with purely classical problem (2.2) we were led to analyze the level crossing problem usually associated with quantum mechanics. The reason is clear that in order to control τ we had to go beyond the diffusion equation (2.1) which does not distinguish between Brownian paths with different exposure times. To solve the clocked Eq. (2.2) we required solutions of Eq. (3.2) for all (complex) values of W as the price for this additional information. Predictions of the theory are easily verified either experimentally, for example, using an ensemble of light sensitive particles, or by trajectory simulations with built-in exposure time control.

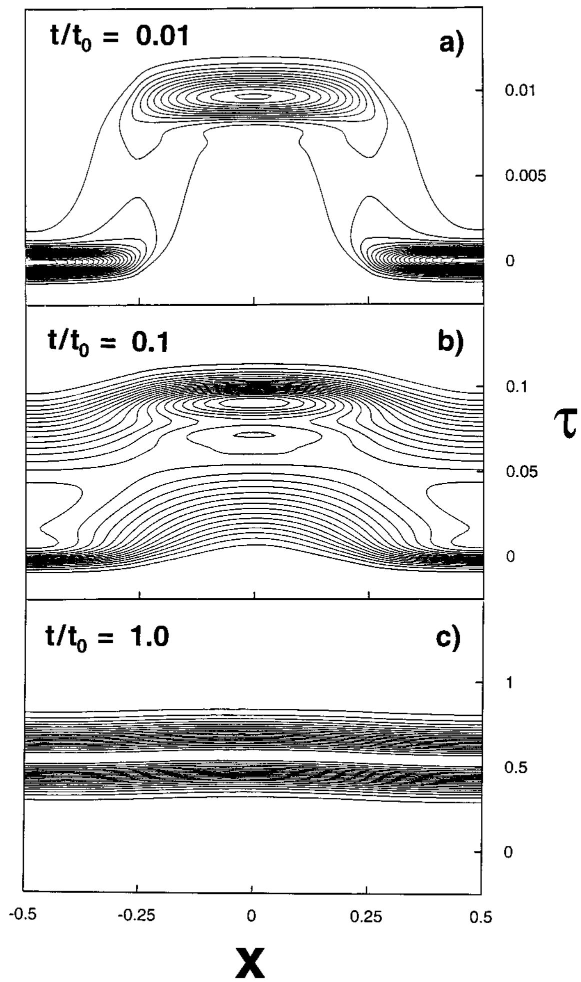

FIG. 4. Contour plots of $Q(x,t|\tau)$ at $t=1$ for (a) $t/t_0 = 0.01$, (b) $t/t_0 = 0.1$, and (c) $t/t_0 = 1.0$, where $t_0 \equiv L^2/D$. The Brownian particles are confined to the interval $[-0.5, 0.5]$, the exposed region is $-0.25 < x < 0.25$. Initial (unnormalized) distribution is a Gaussian, $Q(x,0|\tau) = \exp(-200\tau^2)$.

More generally, we have obtained a general recipe for building in exposure time control into a class of diffusionlike and Boltzmann-like equations by adding to the time derivative a term $\theta_\Omega \partial/\partial\tau$,

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \theta_\Omega \frac{\partial}{\partial \tau}, \tag{7.1}$$

where Ω is some region of the coordinate or phase space. This method has been shown to work also for the nonrelativistic Schrödinger equation [7] and to lead to the so-called Larmor times [1]. It remains to be seen whether meaningful distributions can be obtained for the case electromagnetic wave propagation discussed in Ref. [3].

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